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MULTIBODY SYSTEM DYNAMICS WITH CONSTRAINTS: THE 'CLOSED LOOP' P--ETC(U)
AUG 81 J W KAMMAN, R L HUSTON N00014-76-C-0139
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MULTIBODY SYSTEM DYNAMICS

WITH CONSTRAINTS: THE "CLOSED LOOP" PROBLEM

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1 Aug. 81

Technical Report for Office of Naval Research
Contract N00014-76C-0139

A B S T R A C T

The governing equations for constrained multibody systems are formulated in a manner suitable for their automated, numerical development and solution. Specifically, the "closed loop" problem of multibody chain systems is addressed.

The governing equations are developed by modifying dynamical equations obtained from Lagrange's form of d'Alembert's principle. This modification, which is based upon a solution of the constraint equations obtained through a "zero eigenvalues theorem," is, in effect, a contraction of the dynamical equations.

It is observed that, for a system with n generalized coordinates and m constraint equations, the coefficients in the constraint equations may be viewed as "constraint vectors" in n-dimensional space. Then, in this setting the system itself is free to move in the n-m directions which are "orthogonal" to the constraint vectors.

INTRODUCTION

This report presents a formulation of the governing equations of constrained multibody systems. The objective is the establishment of procedures for the automated generation of the equations.

Recently there has been an increasing interest in the efficient development of governing dynamical equations of multibody systems. This interest is stimulated by the fact that many physical systems can be modelled by systems of connected rigid bodies. Foremost among these physical systems of interest are robots, manipulators, human body models and biodynamic systems, and flexible cables or chains.

There have been a number of formulations of the equations of motion of multibody systems [1-19]*. The majority of these have been restricted to "open chain" or "open tree" systems: that is, systems of rigid bodies such that adjacent bodies have at least one common point and such that no closed loops are formed. Figure 1. illustrates such a system. The formulation of the governing equations of motion of such systems has advanced to the point where the coefficients of the governing differential equations can be formed automatically (numerically) by simply knowing the connection configuration [10-12].

However, during recent years, there has also been interest in the dynamics of systems possessing closed loops, where some of the branches of the tree or chain are connected. Figure 2. illustrates such a system. These systems are useful in modelling such physical systems as: closed

* Numbers in brackets refer to References at the end of the report.

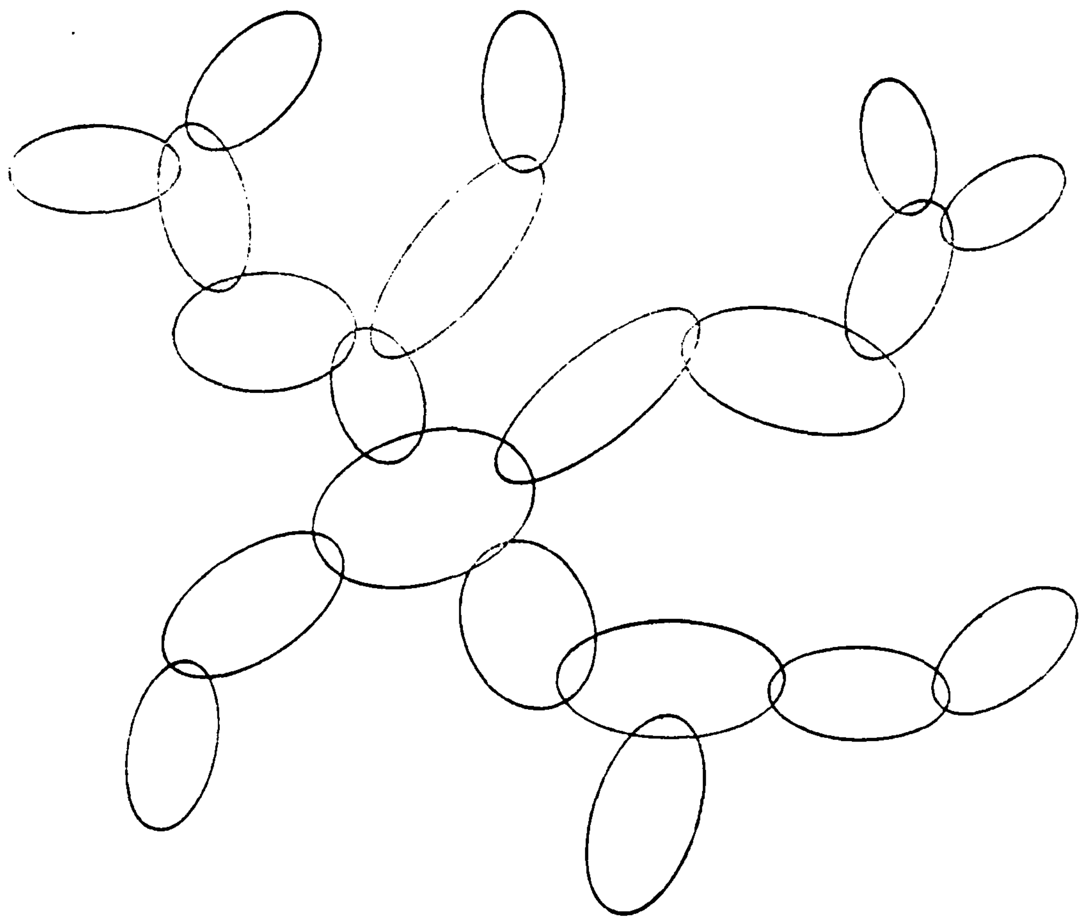

Figure 1. An Open-Chain Multibody System.

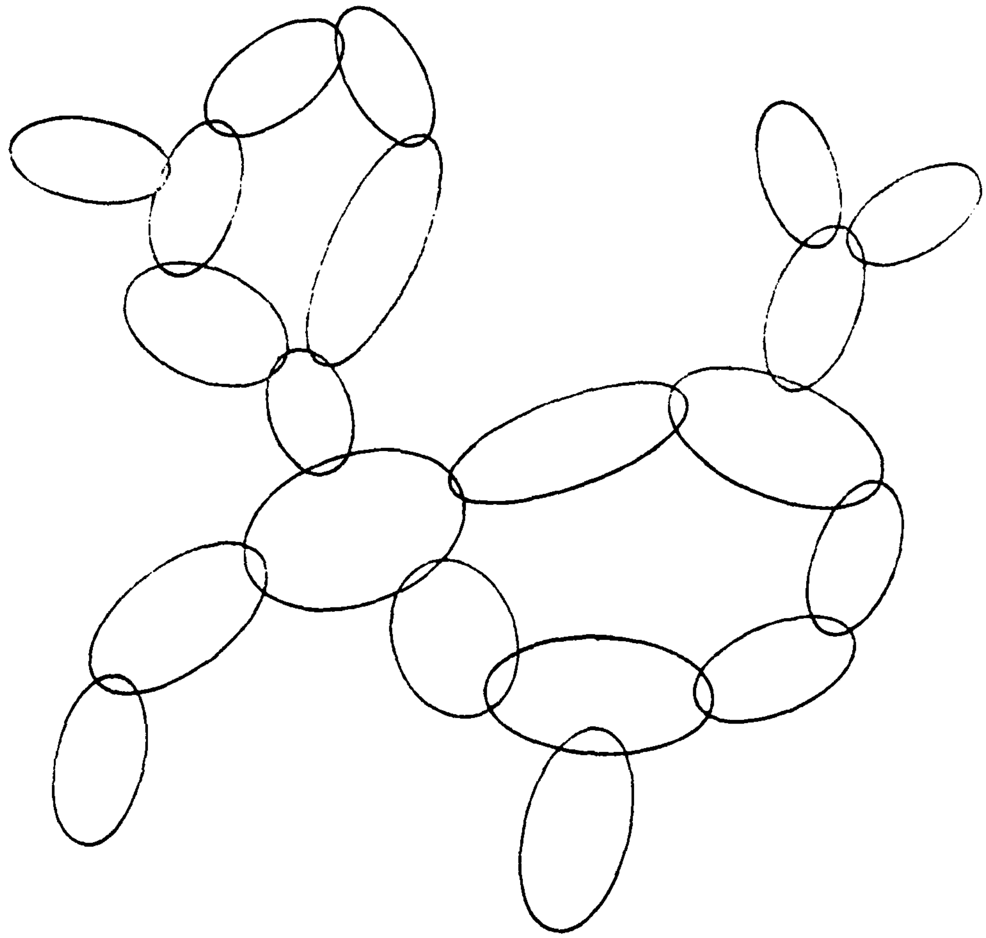

Figure 2. A Multibody Chain System with Closed Loops.

mechanisms, "docking" manipulators of spacecraft, ship cranes, restrained human body models, and cables anchored at both ends.

As noted above, this report presents a procedure for the automatic formulation of the governing equations of such closed-loop multibody systems. The procedure is based upon Lagrange's form of d'Alembert's principle as exposited by Kane et. al. [14,20-22] and as used in [9-12] to develop the dynamical equations of motion. It is also based upon a "zero eigenvalues theorem" as exposited by Walton and Steeves [23] to provide an automatic inclusion in the analysis of the constraint equations. The balance of the report itself is divided into five parts with the following part providing some preliminary information useful in the sequel. This includes a review of dynamical formulations of multibody systems and a statement of the "zero eigenvalues theorem." This is followed in the next part by the governing equation formulation for constrained or closed-loop multibody systems. The subsequent part presents a simple example. The final two parts discuss generalizations and other features of the formulation.

PRELIMINARY CONSIDERATIONS

Coordinates and Kinematics

Consider again the multibody system of Figure 1. This system will have, in general, 3N+3 degrees of freedom where N is the number of bodies of the system. These degrees of freedom might be delineated as follows: Arbitrarily select a body of the system as a reference body. Call this body B_1. Next, label or number the remaining bodies of the system in ascending progression away from B_1 through the branches of the tree structure, moving clockwise from branch to branch. Then the orientation of B_1 relative to a fixed (inertial) reference frame R together with the orientation of the remaining bodies of the system relative to their adjacent lower-numbered bodies defines 3N degrees of freedom. Finally, the location of an arbitrary reference point in B_1 relative to R defines an additional 3 degrees of freedom.

The position and configuration of the system can thus be described by 3N+3 generalized coordinates x_ℓ. Let y_ℓ (ℓ=1,...,3N+3) represent their time derivatives*. Next, let $\underset{\sim}{n}_i$ (i=1,2,3) represent a mutually perpendicular unit vector set fixed in R. Let G_k represent the mass center of body B_k (k=1,...,N). Then, it has been shown [20,21] that the velocity of G_k in R and the angular velocity of B_k in R may be expressed in the form:

$$\underset{\sim}{v}_k = v_{k\ell m} y_\ell \underset{\sim}{n}_m \qquad \text{and} \qquad \underset{\sim}{\omega}_k = \omega_{k\ell m} y_\ell \underset{\sim}{n}_m \tag{1}$$

(Regarding notation, a repeated index, such as ℓ or m in Equation (1) represents a sum over the range of that index, unless otherwise stated.) The coefficients $v_{k\ell m}$ and $\omega_{k\ell m}$ in Equation (1), and their derivatives, play a central role in the analysis of the sequel. They are components of the so-called "partial velocity" and "partial angular velocity" vectors: $\partial\underset{\sim}{v}_k/\partial y_\ell$ and $\partial\underset{\sim}{\omega}_k/\partial y_\ell$. These vectors are useful in forming the generalized

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* The reason for using the symbol y_ℓ instead of $\dot{x}_\ell$ is that there exist instances when a convenient choice of generalized coordinate derivatives result in functions y_ℓ which cannot be integrated to obtain the coordinate x_ℓ. In such cases, the x_ℓ do not, in general, exist (and are sometimes called "quasi-coordinates"). This occurs, for example, when the y_ℓ are selected as angular velocity components. See [24].

active and inertia forces of the system. The coefficients $v_{k\ell m}$ and $\omega_{k\ell m}$ and their derivatives may be formed by simple multiplication algorithms as developed in [9-12]. Hence, by differentiating in Equation (1), the acceleration of G_k in R and the angular acceleration of B_k in R may be expressed as:

$$\underset{\sim}{a}_k = (\dot{v}_{k\ell m} y_\ell + v_{k\ell m} \dot{y}_\ell) \underset{\sim}{n}_m \quad \text{and} \quad \underset{\sim}{\alpha}_k = (\dot{\omega}_{k\ell m} y_\ell + \omega_{k\ell m} \dot{y}_\ell) \underset{\sim}{n}_m \tag{2}$$

Equations of Motion

Consider the system in Figure 1. to be subjected to an externally applied force field which may be represented on a typical body B_k by a single force $\underset{\sim}{F}_k$ passing through G_k together with a couple with torque $\underset{\sim}{M}_k$. Similarly, let the inertia force system on B_k be represented by a single force $\underset{\sim}{F}^*_k$ passing through G_k together with a couple with torque $\underset{\sim}{M}^*_k$. Then $\underset{\sim}{F}^*_k$ and $\underset{\sim}{M}^*_k$ may be expressed as [21]:

$$\underset{\sim}{F}^*_k = -m_k \underset{\sim}{a}_k \quad \text{(no sum)} \tag{3}$$

and

$$\underset{\sim}{M}^*_k = -\underset{\sim}{I}_k \cdot \underset{\sim}{\alpha}_k - \underset{\sim}{\omega}_k \times (\underset{\sim}{I}_k \cdot \underset{\sim}{\omega}_k) \quad \text{(no sum)} \tag{4}$$

where m_k is the mass of B_k and $\underset{\sim}{I}_k$ is the inertia dyadic of B_k relative to G_k. Through use of orthogonal transformation matrices [10] $\underset{\sim}{I}_k$ may be expressed in the form:

$$\underset{\sim}{I}_k = I_{kmn} \underset{\sim}{n}_m \underset{\sim}{n}_n \tag{5}$$

Lagrange's form of d'Alembert's principle then leads to governing dynamical equations of motion of the form [21]:

$$F_\ell + F^*_\ell = 0 \qquad \ell = 1,\ldots,3N+3 \tag{6}$$

where F_ℓ is called the "generalized active force" and may be expressed as:

$$F_\ell = v_{k\ell m} F_{km} + \omega_{k\ell m} M_{km} \tag{7}$$

where there is a sum from 1 to N on k and from 1 to 3 on m, and where F_{km} and M_{km} are the $\underset{\sim}{n}_m$ components of $\underset{\sim}{F}_k$ and $\underset{\sim}{M}_k$. Similarly, F^*_ℓ, in Equation (6), is called the "generalized inertia force" and may be expressed as:

$$F^*_\ell = v_{k\ell m} F^*_{km} + \omega_{k\ell m} M^*_{km} \tag{8}$$

where there is a sum from 1 to N on k and from 1 to 3 on m, and where F^*_{km} and M^*_{km} are the n_m components of $\underset{\sim}{F}^*_k$ and $\underset{\sim}{M}^*_k$.

Substituting from Equations (1) to (5) into (7) and (8) and finally into (6) leads to the equations of motion which may be written in the form[10]:

$$a_{\ell q} \dot{y}_q = f_\ell \qquad \ell = 1,\ldots,3N+3 \tag{9}$$

where there is a sum from 1 to 3N+3 on q and where $a_{\ell q}$ and f_ℓ are given by:

$$a_{\ell q} = m_k v_{k\ell m} v_{kqm} + I_{kmn} \omega_{k\ell m} \omega_{kqm} \tag{10}$$

and

$$f_\ell = F_\ell - (m_k v_{k\ell m} \dot{v}_{kum} y_u + I_{kmn} \omega_{k\ell m} \dot{\omega}_{kun} y_u$$
$$+ e_{nmh} I_{kmr} \omega_{kun} \omega_{ksr} \omega_{k\ell n} y_u y_s) \tag{11}$$

where there is a sum from 1 to N on k, from 1 to 3N+3 on u and s and from 1 to 3 on the other repeated indices and where e_{nmh} is the standard permutation symbol [25].

Constraint Equations

Equations (9) represent the general governing dynamical equations for open chain or open tree systems. However, if the system has one or more closed loops, as illustrated in Figure 2., there are additional equations which need to be satisfied to insure that the closed loops are maintained throughout the motion of the system. These equations are holonomic constraint equations [21] and they may be written in the form:

$$g_i(x_\ell) = 0 \qquad i = 1,\ldots,m : m < 3N+3 \qquad (12)$$

(These equations may be obtained by simply adding to zero the relative position vectors of the connecting joints around the respective loops.) It should be noted that constraint equations of the form of Equation (12) can arise in ways different than that of the closed loops mentioned above. This can occur, for example, with restrictions on the motion at a joint or with the anchoring of one or several of the bodies to a fixed frame R. Finally, by differentiating, Equation (12) becomes a linear relation in the y_ℓ and may be expressed in the form:

$$b_{i\ell} y_\ell = 0 \qquad i = 1,\ldots,m; \quad \ell = 1,\ldots,3N+3 \qquad (13)$$

where the $b_{i\ell}$ are, in general, functions of x_ℓ and t. Equations (9) and (13) thus constitute the governing equations for a "closed-loop" system. These are to be cast into a solvable form in the sequel.

Zero Eigenvalues Theorem

For a constrained N body chain system, the n dynamical equations (9) together with the m constraint equations (13) constitute n+m equations for the n unknown y_ℓ, where n is 3N+3. Hence, the system is over determined. One approach to overcoming this difficulty is to solve Equations (13) for m, say the last m, of the y_ℓ in terms of the first n-m y_ℓ as "independent" generalized coordinate derivatives, the partial velocities and the partial angular velocities can be expressed exclusively in terms of these y_ℓ [21]. Finally, by following the procedure suggested by Equations (6), (7), and (8), n-m governing dynamical equations are obtained for the y_ℓ (ℓ=1,..,n-m).

Although this approach is suitable for relatively small systems, there are difficulties in formulating it for large systems - particularly in attempting to automate it. Among these difficulties is the problem of obtaining a consistent solution of Equations (13) for m of the y_ℓ in terms of the remaining n-m y_ℓ. Another difficulty is the problem of automatically eliminating these m y_ℓ from the partial velocities and the partial angular velocities. However, in 1966, while working on a constraint problem of a different context, Walton and Steeves [23] developed an automated procedure for solving equations such as Equations (13), for m of the y_ℓ in terms of the remaining independent n-m y_ℓ. An extension of their procedure can be developed to automatically eliminate m of the y_ℓ from the partial velocity and partial angular velocity vectors. Their procedure and its extension are based on a "zero eigenvalues theorem" as outlined in the following paragraphs:

Consider Equations (13) to be written in the matrix form as:

$$By = 0 \tag{14}$$

where B is an m x n rectangular matrix with elements $b_{i\ell}$ and y is an n element column matrix with elements v_i. From B form the $n \times n$ symmetric matrix S defined as:

$$S = B^T B \tag{15}$$

where B^T is the transpose of B. Since S is symmetric, there exists an orthogonal matrix T such that:

$$T^T S T = \Lambda \tag{16}$$

where Λ is an $n \times n$ diagonal matrix with <u>real</u> elements or "eigenvalues" λ_i ($i=1,\ldots,n$) [26]. These eigenvalues are readily seen to be <u>non-negative</u> as follows: Let v be a typical column of T and let w be Bv. Then $w^T w = v^T B^T B v = v^T S v$. But $w^T w \geq 0$, and by Equation (16), $v^T S v$ is seen to be an element of Λ, say λ_i. Hence, $\lambda_i \geq 0$. It is also readily seen that there exist <u>zero</u> eigenvalues: Since B is an $m \times n$ matrix, its rank is less than or equal to m [26]. Then, by Equation (15), the rank of S is also less than or equal to m. But, since $m < n$ the rank of the $n \times n$ matrix S is less than n.

Let the columns of T in Equation (16) be arranged so that the eigenvalues of S, or the diagonal elements of Λ, are ordered. That is, arrange T such that $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$. (From the preceeding argument, the last p of these will be zero, where $p \geq n-m$.) Next, let the $m \times n$ matrix D be defined as:

$$D = BT \tag{17}$$

Then, from Equations (15) and (16) it is seen that:

$$D^T D = \Lambda \tag{18}$$

Hence, since the last p rows (and columns) of Λ are zero, C may be written in the partitioned form:

$$D = [\hat{D} | 0] \qquad (19)$$

where $\hat{D}$ is an $(n-p)\times(n-p)$ matrix with mutually orthogonal columns, and where $n-m \leq p < n$. By noting that TT^T is the $n \times n$ identity matrix, the constraint equation (14) may be written as:

$$Dy = DTT^T y \overset{D}{=} DTz = 0 \qquad (20)$$

where z is the n element column matrix defined as $T^T y$. In view of Equation (19), the final equality in Equation (20) is satisfied if the first n-p elements of z are zero, irrespective of the values of the last p elements of z.

Since T is orthogonal, the definition in Equation (20) may be "inverted" leading to the expression:

$$y = Tz \qquad (21)$$

However, since the first n-p elements of z are zero, y may be rewritten as:

$$y = \hat{T}z \qquad (22)$$

where $\hat{T}$ is the $n \times p$ matrix whose columns are the last p columns of T. (In view of the ordering defined above, these columns are the columns of T associated with the zero eigenvalues of S.) Thus, Equation (22) provides a solution to Equation (14) for the n y_i in terms of the p independent (the last p) elements of z. Moreover, Equation (22) is an "algorithmic" expression in that standard numerical procedures exist for matrix diagonalization, eigenvalue determination, and hence, for the numerical evaluation of the n x p matrix $\hat{T}$.

In index notation, Equation (22) may be written as:

$$y_\ell = t_{\ell r} z_r \qquad \ell = 1,\ldots,n;\ r = 1,\ldots,p \tag{23}$$

where the $t_{\ell r}$ may be thought of as components of the column eigenvectors $\underset{\sim}{t}_r$ in n dimensional space.

Finally, the formal statement of Equation (22) constitutes the "zero eigenvalues theorem" [23].

GOVERNING EQUATIONS

The procedures outlined above can be used to systematically formulate the solution to the multibody system dynamics equations (9) subject to the constraint equations (13). To develop this formulation, consider again the partial velocity and partial angular velocity vectors discussed above. From Equations (1) and (23) the velocity of G_k in R and the angular velocity of B_k in R may be expressed in the form:

$$\underset{\sim}{v}_k = v_{k\ell m} t_{\ell r} z_r \underset{\sim}{n}_m \quad \text{and} \quad \underset{\sim}{\omega}_k = \omega_{k\ell m} t_{\ell r} z_r \underset{\sim}{n}_m \tag{24}$$

where the z_r (r=1,...,p) may be viewed as new generalized coordinate derivatives. The partial velocity of G_k in R and the partial angular velocity of B_k in R, with respect to z_r, then become:

$$\partial \underset{\sim}{v}_k/\partial z_r = v_{k\ell m} t_{\ell r} \underset{\sim}{n}_m \quad \text{and} \quad \partial \underset{\sim}{\omega}_k/\partial z_r = \omega_{k\ell m} t_{\ell r} \underset{\sim}{n}_m \tag{25}$$

Hence, the generalized active and inertia forces of Equations (7) and

(8) become:

$$F_r = v_{k\ell m} t_{\ell r} F_{km} + \omega_{k\ell m} t_{\ell r} M_{km} \tag{26}$$

and

$$F^*_r = v_{k\ell m} t_{\ell r} F^*_{km} + \omega_{k\ell m} t_{\ell r} M^*_{km} \tag{27}$$

Then, from Lagrange's form of d'Alembert's principle, the governing equations (6) become:

$$F_r + F^*_r = 0 \qquad r = 1,\ldots,p \tag{28}$$

or, more specifically:

$$a_{\ell q} t_{qr} \dot{y}_\ell = f_\ell t_{\ell r} \qquad r = 1,\ldots,p \tag{29}$$

where $a_{\ell q}$ and f_ℓ are given by Equations (10) and (11).

Equations (29) together with the constraint equations (13) constitute the system of equations to be solved. A numerical procedure for their solution can be formulated as follows: Consider the general case where p=n-m. Then, by differentiating, the constraint equations (13) become:

$$b_{i\ell} \dot{y}_\ell = -\dot{b}_{i\ell} y_\ell \qquad i = 1,\ldots,m \tag{30}$$

Equations (29) and (30) form a total of n equations for the 2n unknowns y_ℓ and x_ℓ. Hence, there needs to be annexed to these equations the expressions:

$$\dot{x}_\ell = y_\ell \qquad \ell = 1,\ldots,n \tag{31}$$

for the consistent numerical formulation of the governing equations. (If the y_ℓ are chosen such that the x_ℓ do not exist, as mentioned earlier, then Equations (11) must be replaced by analogous expressions relating y_ℓ to other variables (such as Euler parameters [10] which define the relative orientations of the bodies.)

The balance of the numerical formulation of the solution of Equations (29), (30), and (31) is now routine: It is perhaps most conveniently expressed in matrix notation. To this end, let C be the n x n matrix containing the coefficients of $\dot{y}_\ell$ in Equations (29) and (30). Then, in partitioned form C is:

$$C = \left[\begin{array}{c} a_{\ell q} t_{qr} \\ \hline b_{i\ell} \end{array} \right] \qquad \begin{array}{l} r = 1,\ldots,n-m \\ i = 1,\ldots,m \\ q,\ell = 1,\ldots,n \end{array} \tag{32}$$

Similarly, let the right sides of Equations (29) and (30) be combined into the column matrix f, which in particioned form is:

$$f = \left[\begin{array}{c} f_\ell t_{\ell r} \\ \hline -b_{i\ell} y_\ell \end{array} \right] \qquad \begin{array}{l} r = 1,\ldots,n-m \\ i = 1,\ldots,m \\ \ell = 1,\ldots,n \end{array} \tag{33}$$

Then the governing equations to be solved may be expressed in the relatively compact matrix form:

$$\dot{y} = C^{-1} f \qquad \text{and} \qquad \dot{x} = y \tag{34}$$

where x and y are the column matrices with elements x_ℓ and y_ℓ ($\ell = 1,\ldots,n$) respectively.

EXAMPLE

For a simple example illustrating some of these ideas, consider the planar triple pendulum shown in Figure 3. The three rods are identical having length ℓ, and there are frictionless pins at the joints: O_1, O_2, and O_3. The system has 3 degrees of freedom which may be described by the orientation angles θ_1, θ_2, and θ_3 shown in the figure. Using Lagrange's form of d'Alembert's principle, the equations of motion of the system take the form:

$$
\begin{aligned}
&(24 + 18C_2 + 6C_3 + 6C_{2+3})\ddot{\theta}_1 + (10 + 9C_2 + 6C_3 + 3C_{2+3})\ddot{\theta}_2 \\
&+ (2 + 3C_3 + 3C_{2+3})\ddot{\theta}_3 = -(g/\ell)(15S_1 + 9S_{1+2} + 3S_{1+2+3}) \\
&- (9S_2 + 3S_{2+3})\dot{\theta}_1^2 + (9S_2 - 3S_3)(\dot{\theta}_1 + \dot{\theta}_2)^2 \\
&+ (3S_{2+3} + 3S_3)(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2 \qquad (35)
\end{aligned}
$$

$$
\begin{aligned}
&(10 + 9C_2 + 6C_3 + 3C_{2+3})\ddot{\theta}_1 + (10 + 6C_3)\ddot{\theta}_2 + (2 + 3C_3)\ddot{\theta}_3 \\
&= -(g/\ell)(9S_{1+2} + 3S_{1+2+3}) - (9S_2 + 3S_{2+3})\dot{\theta}_1^2 \\
&- 3S_3(\dot{\theta}_1 + \dot{\theta}_2)^2 + 3S_3(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2 \qquad (36)
\end{aligned}
$$

and

$$
\begin{aligned}
&(2 + 3C_3 + 3C_{2+3})\ddot{\theta}_1 + (2 + 3C_3)\ddot{\theta}_2 + 2\ddot{\theta}_3 \\
&= 3(g/\ell)S_{1+2+3} - 3S_{2+3}\dot{\theta}_1^2 - 3S_3(\dot{\theta}_1 + \dot{\theta}_2)^2 \qquad (37)
\end{aligned}
$$

where $C_i = \cos\theta_i$, $C_{i+j} = \cos(\theta_i + \theta_j)$, etc.

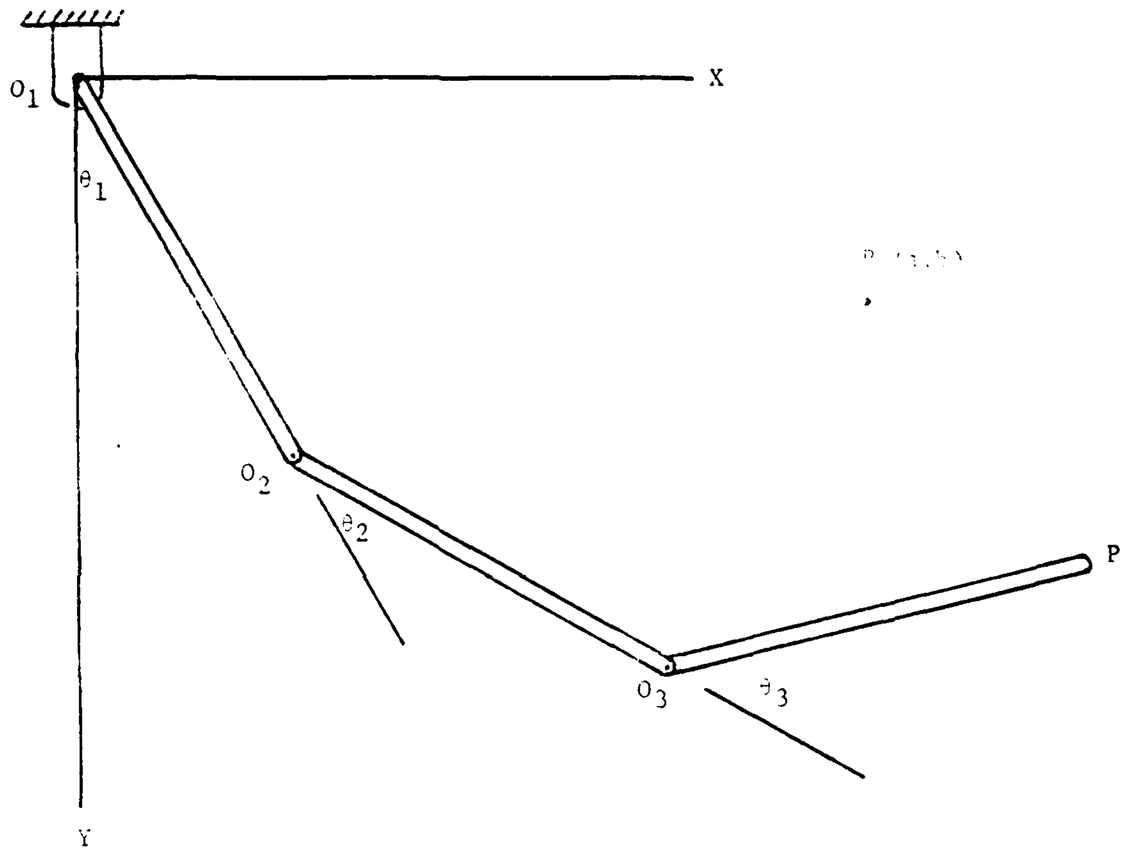

Figure 3. Planar Triple Pendulum.

A "closed loop" or constraint may be formed by fixing the end point P of the pendulum. Hence, let P be fixed at a point P_o having coordinates (a,b) relative to the X-Y coordinate system shown in Figure 3. This constrained system has only one degree of freedom. Two scalar constraint equations relating the coordinates θ_1, θ_2, and θ_3 may be obtained from the position vector equation:

$$\underset{\sim}{O_1O_2} + \underset{\sim}{O_2O_3} + \underset{\sim}{O_3P_o} + \underset{\sim}{P_oO_1} = 0 \tag{38}$$

That is, considering the horizontal and vertical components of this equation leads to the equations:

$$\begin{aligned} S_1 + S_{1+2} + S_{1+2+3} &= a/\ell \\ C_1 + C_{1+2} + C_{1+2+3} &= b/\ell \end{aligned} \tag{39}$$

which, upon differentiation, become:

$$\begin{aligned} (C_1 + C_{1+2} + C_{1+2+3})\dot{\theta}_1 + (C_{1+2} + C_{1+2+3})\dot{\theta}_2 + C_{1+2+3}\dot{\theta}_3 &= 0 \\ (S_1 + S_{1+2} + S_{1+2+3})\dot{\theta}_1 + (S_{1+2} + S_{1+2+3})\dot{\theta}_2 + S_{1+2+3}\dot{\theta}_3 &= 0 \end{aligned} \tag{40}$$

Equations (39) and (40) represent Equations (12) and (13) in the foregoing analysis.

To simplify the analysis, let P be fixed on the X-axis at a point P_o a distance ℓ from O_1. The system then takes the form of a rhombic linkage as shown in Figure 4. In this case, $a=\ell$, $b=0$, and the constraint equations (39) are seen to be satisfied by the relations: $\theta_2 = \pi/2-\theta_1$ and $\theta_3 = \pi/2+\theta_1$. The coefficient matrix B of Equations (13) and (40) may then be expressed as:

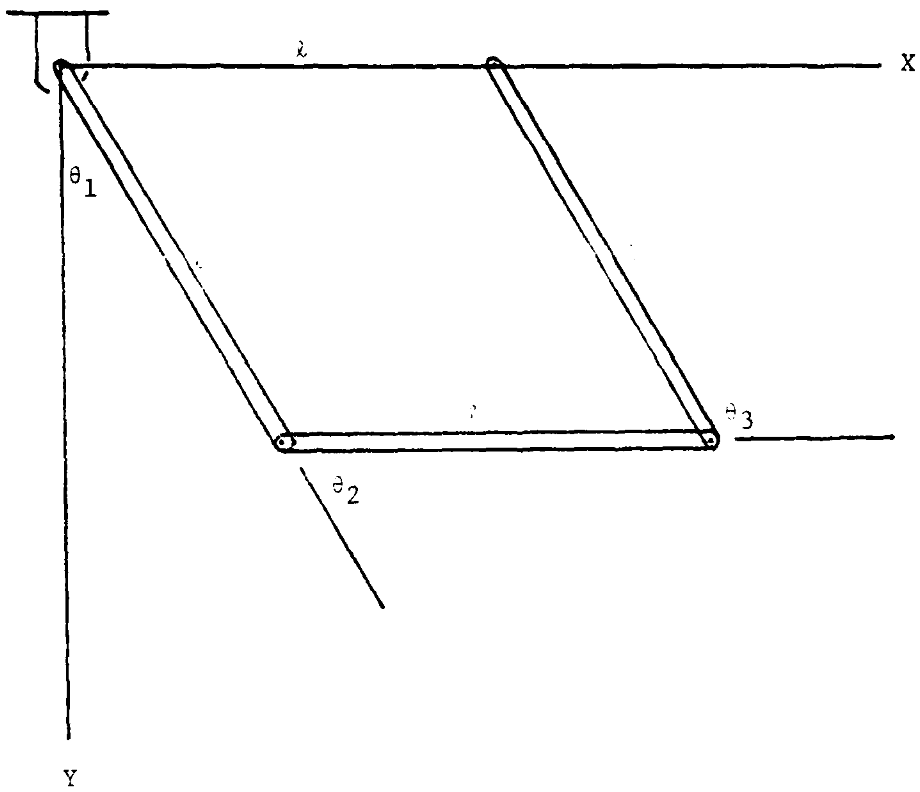

Figure 4. Constrained Triple Pendulum.

$$B = \begin{bmatrix} 0 & -C_1 & -C_1 \\ 1 & (1 - S_1) & -S_1 \end{bmatrix} \tag{41}$$

The matrix S of Equation (15) then becomes:

$$S = \begin{bmatrix} 1 & (1 - S_1) & -S_1 \\ (1 - S_1) & 2(1 - S_1) & (1 - S_1) \\ -S_1 & (1 - S_1) & 1 \end{bmatrix} \tag{42}$$

It is readily seen that S has one zero eigenvalue and that the associated eigenvector array $\hat{T}$ is:

$$\hat{T} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \tag{43}$$

The governing equations to be solved may now be obtained using Equations (29) and (30). From Equations (35), (36), (37), and (40), these become:

$$\begin{aligned} &(16 + 9C_2 + 3C_3 + 6C_{2+3})\ddot{\theta}_1 + (2 + 9C_2 + 3C_3 + 3C_{2+3})\ddot{\theta}_2 \\ &+ (2 + 3C_{2+3})\ddot{\theta}_3 = -(g/\ell)(15S_1 + 3S_{1+2+3}) - 3S_{2+3}\dot{\theta}_1^2 \\ &+ (9S_2 - 3S_3)(\dot{\theta}_1 + \dot{\theta}_2)^2 + 3S_{2+3}(\dot{\theta}_1 + \dot{\theta}_2 + \dot{\theta}_3)^2 \end{aligned} \tag{44}$$

$$-C_1\ddot{\theta}_2 - C_1\ddot{\theta}_3 = S_1\dot{\theta}_1\dot{\theta}_2 + S_1\dot{\theta}_1\dot{\theta}_3 \tag{45}$$

and

$$\ddot{\theta}_1 + (1 - S_1)\ddot{\theta}_2 - S_1\ddot{\theta}_3 = -C_1\dot{\theta}_1\dot{\theta}_2 - C_1\dot{\theta}_1\dot{\theta}_3 \tag{46}$$

Equations (44), (45), and (46) have been solved numerically. The results match the results of the numerical solution of the pendulum equation:

$$\ddot{\theta}_1 + (6g/5\ell)\sin\theta_1 = 0 \tag{47}$$

which is the governing equation of the rhombic linkage of Figure 4.

REDUCTION OF THE NUMBER OF GOVERNING EQUATIONS

A principal step in the foregoing formulation as well as in the above example is the differentiation of the constraint equations (See Equations (36), (45), and (46).). Since these equations are then combined with the reduced set of dynamical equations and the resulting system is integrated, a question which arises is: Is it necessary to first differentiate and later integrate these equations? That is, could some computational efficiency be obtained by avoiding the differentiation-integration steps? The answer to these questions is that it is indeed possible to integrate fewer equations and thus obtain some computational advantages. A procedure demonstrating this is outlined in the following paragraphs:

Let the $m \times n$ matrix B in Equation (14) be partitioned into two submatrices as:

$$B = [B_a \vdots B_b] \tag{48}$$

where B_a is an $m \times (n-m)$ array and B_b is an $m \times m$ square array. This partitioning of B induces a partitioning of the $\dot{y}$ array in Equation (30).

That is, if Equation (30) is written in the matrix form as: $B\dot{y} = -\dot{B}y$, then $B\dot{y}$ may be expressed as:

$$B\dot{y} = [B_a \vdots B_b] \begin{bmatrix} \dot{y}_a \\ \hline \dot{y}_b \end{bmatrix} = -\dot{B}y \tag{49}$$

where $\dot{y}_a$ is a column array with n-m elements and $\dot{y}_b$ is a column array with m elements. If B_b^{-1} exists, Equation (49) may be solved for $\dot{y}_b$ in the form:

$$\dot{y}_b = -B_b^{-1}(B_a\dot{y}_a + \dot{B}y) \tag{50}$$

Similarly, let the reduced set of dynamical equations (29) be written in the matrix form $\hat{A}\dot{y} = \hat{f}$ where $\hat{A}$ is the (n-m) x m array with elements $t_{rq}a_{q\ell}$; $r=1,\ldots,n-m$; $\ell=1,\ldots,n$ and $\hat{f}$ is the column array with elements $f_q t_{qr}$ $r=1,\ldots,n-m$. Then following the pattern of Equations (48) and (49), let the dynamical equations be written in the partitioned form:

$$[\hat{A}_a \vdots \hat{A}_b] \begin{bmatrix} \dot{y}_a \\ \hline \dot{y}_b \end{bmatrix} = \hat{f} \tag{51}$$

where $\hat{A}_a$ is an (n-m)x(n-m) square array and $\hat{A}_b$ is an (n-m) x m array.

By substituting from Equations (50) and by matrix block multiplication, Equation (51) becomes:

$$\hat{A}_a\dot{y}_a + \hat{A}_b[-B_b^{-1}(B_a\dot{y}_a + \dot{B}y)] = \hat{f} \tag{52}$$

By rearranging the terms, this equation may be written as:

$$\hat{\hat{A}}\dot{y}_a = \hat{\hat{f}} \tag{53}$$

where $\hat{\hat{A}}$ is the (n-m) x (n-m) square array defined as:

$$\hat{\hat{A}} = \hat{A}_a - \grave{A}_b B_b^{-1} B_a \tag{54}$$

and $\hat{\hat{f}}$ is the (n-m) element column array defined as:

$$\hat{\hat{f}} = \hat{f} + \grave{A}_b B_b^{-1} \dot{B} y \tag{55}$$

Equation (53) is equivalent to a system of (n-m) scalar differential equations containing 2n unknowns: y_ℓ and x_ℓ, $\ell=1,\ldots,n$. Hence, there needs to be annexed to this system an additional n+m scalar equations. Equation (31) provides n of these equations. In matrix form they may be written as:

$$\dot{x} = y \tag{56}$$

The final m equations may be obtained from Equations (50). However, unlike Equations (53) and (56), Equations (50) are <u>algebraic</u> equations and do not need to be integrated. That is, the system of 2n equations of Equations (50), (53), and (56) contain 2n-m differential equations and m algebraic equations for the 2n unknowns y_ℓ and x_ℓ, $\ell=1,\ldots,n$. This is a reduction of m differential equations from the previous system of Equations (29), (30), and (31).

DISCUSSION

At this point there are several comments and observations which might be helpful. First, in the procedure of the zero eigenvalues theorem, the m constraint equations are solved for the n y_ℓ in terms of n-m new variables z_r. Interestingly, in the subsequent formulation of governing equations, these new variables z_r do not appear. Indeed, it is only the coefficients

$t_{\ell r}$ of the z_r which are used. As noted earlier, these coefficients are the components in n-dimensional space of the eigenvectors $\underset{\sim}{t}_r$ associated with the zero eigenvalues of S. However, in this context, since the corresponding eigenvalues are zero, $S\underset{\sim}{t}_r$ is zero and the eigenvectors $\underset{\sim}{t}_r$ are thus "orthogonal" to the rows of S. This in turn means that these eigenvectors are orthogonal to the rows of the constraint matrix B. (This conclusion was also reached in an earlier analysis of constraint equations in n-dimensional space [27].) Hence, let the rows of B be thought of as "constraint vectors" in n-dimensional space. Then, since the $\underset{\sim}{t}_r$ are used to form the new partial velocity and partial angular velocity vectors, the physical system can be considered to be constrained to move, in n-dimensional space, in directions orthogonal to these constraint vectors -- that is, in directions defined by the eigenvectors $\underset{\sim}{t}_r$.

Next, Lagrange's form of d'Alembert's principle is an ideally suited method for formulating the dynamical equations when there are accompaning constraint equations. Indeed, the governing differential equations may be developed by simply contracting the dynamical equations obtained, via the principle, by using the $t_{\ell k}$ array obtained from the zero eigenvalues theorem. This procedure is seen to be successful since the generalized forces are linear, homogeneous functions of the partial velocity and angular velocity vectors, which in turn, are coefficients of the generalized coordinate derivatives (in the velocity and angular velocity vectors). Therefore, a modification of these derivatives directly changes these vectors and hence, also the generalized forces. This means that the modification procedure for the generalized coordinate derivatives, as developed by the zero eigenvalues theorem, may be directly applied to the dynamical equations themselves. Also, due to these arguments, it is seen that this procedure would not necessarily be successful if the dynamical equations were obtained by some other method. (Additional discussion of the merits of Lagrange's form of d'Alembert's principle may be found in References [10,14,20,21,22].)

Finally, the procedure developed herein is deemed to be well suited for the automated development of the governing equations. Numerical algorithms are currently being prepared to be incorporated into the computer codes discussed in [10,11,12]. Additional information on this may be obtained from the authors.

ACKNOWLEDGEMENT

The authors gratefully acknowledge partial support for this research from the Office of Naval Research under Contract N00014-76C-0139. The authors also gratefully acknowledge several helpful suggestions of Dr. Chris Passerello of Michigan Technological University.

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SECURITY CLASSIFICATION OF THIS PAGE

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: ONR-UC-MIE-080181-12 — 2. GOVT ACCESSION NO.: AD-A103081	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): Multibody System Dynamics with Constraints: The "Closed Loop" Problem	5. TYPE OF REPORT & PERIOD COVERED: Technical 7/31/80-8/1/81
	6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): James W. Kamman, Ronald L. Huston	8. CONTRACT OR GRANT NUMBER(s): N00014-76C-0139
9. PERFORMING ORGANIZATION NAME AND ADDRESS: University of Cincinnati, Cincinnati, Ohio 45221	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 122303
11. CONTROLLING OFFICE NAME AND ADDRESS: ONR Resident Research Representative, Ohio State University, 1314 Kenner Rd., Columbus, OH 43212	12. REPORT DATE: 8/1/81
	13. NUMBER OF PAGES: 26
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office): Office of Naval Research, Structural Mechanics Code, Department of the Navy, Arlington, VA 22217	15. SECURITY CLASS. (of this report): Unclassified
	15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Distribution of this report is unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Constraint Equations, Finite-Segment Modelling, Closed Chains, Dynamics, Multi-Body Systems.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

The governing equations for constrained multibody systems are formulated in a manner suitable for their automated, numerical development and solution. Specifically, the "closed loop" problem of multibody chain systems is addressed.

The governing equations are developed by modifying dynamical equations obtained from Lagrange's form of d'Alembert's principle. This modification, which is based upon a solution of the constraint equations obtained through a "zero eigenvalues theorem," is, in effect, a contraction of the dynamical equations.

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20. ABSTRACT (Continued)

It is observed that, for a system with n generalized coordinates and m constraint equations, the coefficients in the constraint equations may be viewed as "constraint vectors" in n-dimensional space. Then, in this setting the system itself is free to move in the n-m directions which are "orthogonal" to the constraint vectors.